Magnetic Reduction of Regular Controlled Hamiltonian System with Symmetry of the Heisenberg Group

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Dedicated to Professor Tudor S. Ratiu on the occasion of his 65th birthday
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Abstract: In this paper, we study the regular point reduction of regular controlled Hamiltonian system with magnetic symplectic form and the symmetry of the Heisenberg group. We give the reduced regular controlled Hamiltonian system on the generalization of coadjoint orbit of the Heisenberg group by calculation in detail, and discuss the magnetic reducible controlled Hamiltonian (MRCH) equivalence. As an application, we consider the motion of the Heisenberg particle in a magnetic field.

Keywords: Heisenberg group, regular controlled Hamiltonian system, regular point reduction, magnetic symplectic form, MRCH-equivalence.

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Contents

1 Introduction

The reduction theory for mechanical system with symmetry has its origin in the classical work of Euler, Lagrange, Hamilton, Jacobi, Routh, Liouville and Poincaré and its modern geometric formulation in the general context of symplectic manifolds and equivariant momentum maps is

developed by Meyer, Marsden and Weinstein; see Abraham and Marsden [1] or Marsden and Weinstein [11] and Meyer [12]. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. Also see Abraham et al. [2], Arnold [3], Libermann and Marle [4], Marsden [5], Marsden et al. [6], Marsden and Ratiu [9] and Ortega and Ratiu [13] for more details.

The reduction theory of cotangent bundle of a configuration manifold is a very important special case of general reduction theory. We first give a precise analysis of the geometrical structure of symplectic reduced space of a cotangent bundle. Let Q be a smooth manifold and TQ the tangent bundle, T^*Q the cotangent bundle with a canonical symplectic form ω_0. Assume that $\Phi : G \times Q \to Q$ is a left smooth action of a Lie group G on the manifold Q. The cotangent lift is the action of G on T^*Q, $\Phi^{T^*} : G \times T^*Q \to T^*Q$ given by $g \cdot \alpha_q = (T\Phi_{g^{-1}})^* \cdot \alpha_q, \ \forall \ \alpha_q \in T^*_qQ, \ q \in Q$. The cotangent lift of any proper (resp. free) G-action is proper (resp. free). Each cotangent lift action is symplectic with respect to the canonical symplectic form ω_0, and has an Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by $< \mathbf{J}(\alpha_q), \xi >= \alpha_q(\xi_Q(q))$, where $\xi \in \mathfrak{g}$, $\xi_Q(q)$ is the value of the infinitesimal generator ξ_Q of the G-action at $q \in Q$, $<,>: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the duality pairing between the dual $\mathfrak{g}^*$ and $\mathfrak{g}$. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $\mathbf{J}$, and $G_\mu = \{g \in G | \ \mathrm{Ad}^*_g \mu = \mu\}$ is the isotropy subgroup of the coadjoint G-action at the point μ. From Marsden et al. [6], we know that the classification of symplectic reduced space of a cotangent bundle as follows. (1) If $\mu = 0$, the symplectic reduced space of cotangent bundle T^*Q at $\mu = 0$ is given by $((T^*Q)_\mu, \omega_\mu) = (T^*(Q/G), \omega_0)$, where ω_0 is the canonical symplectic form of cotangent bundle $T^*(Q/G)$. Thus, the symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ at $\mu = 0$ is a symplectic vector bundle. (2) If $\mu \neq 0$, and G is Abelian, then $G_\mu = G$, in this case the regular point symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to symplectic vector bundle $(T^*(Q/G), \omega_0 - B_\mu)$, where B_μ is a magnetic term. (3) If $\mu \neq 0$, and G is not Abelian and $G_\mu \neq G$, in this case the regular point symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle over $T^*(Q/G_\mu)$ with fiber to be the coadjoint orbit $\mathcal{O}_\mu$, see the cotangent bundle reduction theorem—bundle version, also see Marsden and Perlmutter [8]. Thus, from the above discussion, we know that the symplectic reduced space of a cotangent bundle may not be a cotangent bundle.

On the other hand, in mechanics, the phase space of a Hamiltonian system is very often the cotangent bundle T^*Q of a configuration manifold Q. Therefore, the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle T^*Q may not be a Hamiltonian system on a cotangent bundle, that is, the set of Hamiltonian systems with symmetry on the cotangent bundle is not complete under the regular symplectic reduction. We also note that, in the case of regular orbit reduction, from Ortega and Ratiu [13] and the regular reduction diagram, we know that the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$, and hence is symplectically diffeomorphic to a symplectic fiber bundle. If we may define a regular controlled Hamiltonian (RCH) system on a symplectic fiber bundle, then it is possible to describe uniformly the RCH systems on T^*Q and their regular reduced RCH systems on the associated reduced spaces. This is why the authors in Marsden et al. [10] set up the regular reduction theory of RCH systems on a symplectic fiber bundle, by using momentum map and the associated reduced symplectic

forms and from the viewpoint of completeness of regular symplectic reduction. In addition, a good expression of the dynamical vector field of RCH system is given in Marsden et al. [10], such that we can describe the feedback control law to modify the structure of RCH system and discuss RCH-equivalence.

In the following we introduce some generalizations around the above work in Marsden et al. [10]. At first, it is a natural problem what and how we could do, if we define a controlled Hamiltonian system on the cotangent bundle T^*Q by using a Poisson structure, and if symplectic reduction procedure given by Marsden et al. [10] does not work or is not efficient enough. In Wang and Zhang [19], we set up the optimal reduction theory of a CH systems with Poisson structure and symmetry by using the optimal momentum map and reduced Poisson tensor (resp. reduced symplectic form), and discuss the CH-equivalence. Next, it is worthy of note that if there is no momentum map for our considered systems, then the reduction procedures given in Marsden et al. [10] and Wang and Zhang [19] can not work. One must look for a new way. On the other hand, motivated by the work of Poisson reduction by distribution for a Poisson manifold, we note that in above research work, the phase space T^*Q of the CH system is a Poisson manifold, and its control subset $W \subset T^*Q$ is a fiber submanifold. If we assume that $D \subset TT^*Q|_W$ is a controllability distribution of the CH system, then we can study the Poisson reduction for the CH system by controllability distribution, and the relationship between Poisson reduction for the regular (resp. singular) Poisson reducible CH systems by G-invariant controllability distributions and that for the associated reduced CH systems by the reduced controllability distributions. See Ratiu and Wang [14] for more details. In addition, as the applications, we also study the underwater vehicle-rotors system and rigid spacecraft-rotors system, as well as Hamilton-Jacobi theory for the RCH system with symmetry, which show the effect on regular symplectic reduction of RCH system. See Wang [15–18] for more details. These research works not only gave a variety of reduction methods for controlled Hamiltonian systems, but also showed a variety of relationships of the controlled Hamiltonian equivalences of these systems.

The theory of controlled mechanical systems is a very important subject, following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied. We know that it is not easy to give the precise analysis of geometrical and topological structures of the phase spaces and the reduced phase spaces of various Hamiltonian systems. The study of completeness of Hamiltonian reductions for controlled Hamiltonian system with symmetry is related to the geometrical structures of Lie group, configuration manifold and its cotangent bundle, as well as the action ways of Lie group on the configuration manifold and its cotangent bundle. Our goal to do the research is to set up the various perfect reduction theory for controlled mechanical systems, along the ideas of Professor Jerrold E. Marsden, by analyzing carefully the geometrical and topological structures of the phase spaces of various mechanical systems.

Recently, we note that the Heisenberg group is an important Lie group and it is a central extension of $\mathbb{R}^2$ by $\mathbb{R}$, and hence it is also a motivating example for the general theory of Hamiltonian reduction by stages, see Marsden et al. [6, 7]. In particular, we note that there is a magnetic term on the cotangent bundle of the Heisenberg group $\mathcal{H}$, which is related to a curvature two-form of a mechanical connection determined by the reduction of center action of the Heisenberg group $\mathcal{H}$, see Theorem 3.2 in §3, such that we can study the magnetic reduction

of the cotangent bundle of the Heisenberg group $\mathcal{H}$ and the RCH system with symmetry of the Heisenberg group $\mathcal{H}$. These are the main works in this paper. It is worthy of note that the magnetic reduction is not the reduction of canonical structure of cotangent bundle, it reveals the deeper relationship of intrinsic geometrical structures of RCH systems.

A brief of outline of this paper is as follows. In the second section, we review some relevant definitions and basic facts about the Heisenberg group $\mathcal{H}$, coadjoint $\mathcal{H}$-action and coadjoint orbit, which will be used in subsequent sections. The magnetic reduction of the cotangent bundle of the Heisenberg group $\mathcal{H}$, and the magnetic term in the magnetic symplectic form of cotangent bundle of the Heisenberg group $\mathcal{H}$, which is related to a curvature two-form of a mechanical connection determined by the reduction of center action of the Heisenberg group $\mathcal{H}$, are introduced in the third section. In the fourth section, we introduce briefly some relevant definitions and basic facts about the RCH systems defined on a symplectic fiber bundle and on the cotangent bundle of a configuration manifold, respectively, and RCH-equivalence, the regular point reducible RCH system with symmetry, the RpCH-equivalence, and give the regular point reduction theorem. In the fifth section, we discuss the magnetic reduction of a RCH system with symmetry of the Heisenberg group $\mathcal{H}$ and MRCH-equivalence, and give the reduced regular controlled Hamiltonian system on the generalization of coadjoint orbit of the Heisenberg group by calculation in detail. Moreover, we prove the magnetic reduction theorem for the RCH system with symmetry of the Heisenberg group, which explains the relationship between MRCH-equivalence for the magnetic point reducible RCH systems with symmetry of the Heisenberg group and RCH-equivalence for the associated magnetic point reduced RCH systems. As an application of the theoretical results, in the sixth section, we consider the motion of the Heisenberg particle in a magnetic field, and we also consider the magnetic term from the viewpoint of Kaluza-Klein construction. These research works develop the theory of Hamiltonian reduction for the RCH system with symmetry of the Heisenberg group and make us have much deeper understanding and recognition for the geometrical structures of controlled Hamiltonian systems.

2 The Heisenberg Group and Its Lie Algebra

In this paper, our goal is to give the magnetic reduction of the cotangent bundle of the Heisenberg group $\mathcal{H}$ and the RCH system with symmetry of the Heisenberg group $\mathcal{H}$. In order to do these, in this section, we first review some relevant definitions and basic facts about the Heisenberg group $\mathcal{H}$, coadjoint $\mathcal{H}$-action and coadjoint orbit, which will be used in subsequent sections. We shall follow the notations and conventions introduced in Marsden et al. [6, 7].

We consider the commutative group $\mathbb{R}^2$ with its standard symplectic form ω, which is the usual area form on the Euclidean plane, that is,

$$\omega(u, v) = u_1 v_2 - u_2 v_1, \tag{2.1}$$

where $u = (u_1, u_2),\ v = (v_1, v_2) \in \mathbb{R}^2$. Define the set $\mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R}$ with group multiplication

$$(u, \alpha)(v, \beta) = (u + v, \alpha + \beta + \frac{1}{2}\omega(u, v)) \tag{2.2}$$

where $u,\ v \in \mathbb{R}^2$ and $\alpha,\ \beta \in \mathbb{R}$. It is readily verified that this operation defines a Lie group, and its identity element is $(0, 0)$ and the inverse of (u, α) is given by $(u, \alpha)^{-1} = (-u, -\alpha)$. This

group is called the **Heisenberg group**, which is an important Lie group and it is isomorphic to the upper triangular 3×3 matrices with ones on the diagonal, and the isomorphism is given by

$$(u, \alpha) \mapsto \begin{bmatrix} 1 & u_1 & \alpha + \frac{1}{2}u_1 u_2 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

It is worthy of note that each element $(0, \alpha)$ in $\mathcal{H}$ commutes with every other element of $\mathcal{H}$, and by using the nondegeneracy of the symplectic form ω, we know that every element of $\mathcal{H}$ that commutes with all other elements of $\mathcal{H}$ is of the form $(0, \alpha)$. Hence, the subgroup $A = \{(0, \alpha) \in \mathcal{H} | \alpha \in \mathbb{R}\}$, consisting of pairs $(0, \alpha)$ in $\mathcal{H}$, is the **center** of $\mathcal{H}$ and $A \cong \mathbb{R}$. Thus, the Heisenberg group $\mathcal{H}$ is the **central extension** of $\mathbb{R}^2$ by $\mathbb{R}$ and $B = \omega$ is its group two-cocycle, see Marsden et al. [6].

The Lie algebra of the Heisenberg group $\mathcal{H}$ is $\eta \cong \mathbb{R}^2 \oplus \mathbb{R}$. We identify η with $\mathbb{R}^3$ via the Euclidean inner product. In the following we can calculate the Lie algebra bracket on η. We know that the left and right translation on $\mathcal{H}$ induce the left and right action of $\mathcal{H}$ on itself. The conjugation action $I : \mathcal{H} \to \mathcal{H}$ is given by

$$I_{(u,\alpha)}((v, \beta)) = (u, \alpha)(v, \beta)(u, \alpha)^{-1} = (u, \alpha)(v, \beta)(-u, -\alpha) = (v, \beta + \omega(u, v)), \tag{2.4}$$

for any $(u, \alpha),\ (v, \beta) \in \mathcal{H}$, which is the inner automorphism on $\mathcal{H}$. By differentiating the above formula with respect to (v, β), we can see that the operator $\mathrm{Ad} : \mathcal{H} \times \eta \to \eta$, which induces an adjoint action of $\mathcal{H}$ on η, is given by

$$\mathrm{Ad}_{(u,\alpha)}(Y, b) = (Y, b + \omega(u, Y)), \tag{2.5}$$

where $(Y, b) \in \eta$. By differentiating once more the above formula with respect to (u, α), we can get that the operator $\mathrm{ad} : \eta \times \eta \to \eta$, that is, the Lie bracket $[,] : \eta \times \eta \to \eta$, is given by

$$\mathrm{ad}_{(X,a)}(Y, b) = [(X, a), (Y, b)] = (0, \omega(X, Y)), \tag{2.6}$$

with a Lie algebra two-cocycle $C(X, Y) = \omega(X, Y)$, where $(X, a),\ (Y, b) \in \eta \cong \mathbb{R}^2 \oplus \mathbb{R}$.

The dual of Lie algebra η of the Heisenberg group $\mathcal{H}$ is $\eta^* \cong \mathbb{R}^2 \oplus \mathbb{R}$. We also identify η^* with $\mathbb{R}^3$ via the Euclidean inner product. Note that the adjoint representation of the Heisenberg group $\mathcal{H}$ is defined by $\mathrm{Ad}_{(u,\alpha)}(Y, b) = T_{(0,0)}I_{(u,\alpha)}(Y, b) = T_{(u,\alpha)^{-1}}L_{(u,\alpha)} \cdot T_{(0,0)}R_{(u,\alpha)^{-1}}(Y, b) : \mathcal{H} \times \eta \to \eta$, then the coadjoint representation of the Heisenberg group $\mathcal{H}$, that is, $\mathrm{Ad}^* : \mathcal{H} \times \eta^* \to \eta^*$, is defined by the following equation

$$\langle \mathrm{Ad}^*_{(u,\alpha)^{-1}}(\mu, \nu), (Y, b)\rangle = \langle (\mu, \nu), \mathrm{Ad}_{(u,\alpha)}(Y, b)\rangle, \tag{2.7}$$

where $(u, \alpha) \in \mathcal{H},\ (\mu, \nu) \in \eta^*$, and $(Y, b) \in \eta$, and $\langle, \rangle$ denotes the natural pairing between η^* and η. Moreover, the coadjoint representation $\mathrm{Ad}^* : \mathcal{H} \times \eta^* \to \eta^*$ induces a left coadjoint action of the Heisenberg group $\mathcal{H}$ on η^*, which is given by

$$\mathrm{Ad}^*_{(u,\alpha)^{-1}}(\mu, \nu) = (\mu + \nu \mathbb{J}u, \nu), \tag{2.8}$$

where $(u, \alpha) \in \mathcal{H},\ (\mu, \nu) \in \eta^*$, and $\mathbb{J}u = \mathbb{J}(u_1, u_2) = (u_2, -u_1)$ is the matrix of the standard symplectic form ω on $\mathbb{R}^2$. Then the coadjoint orbit $\mathcal{O}_{(\mu,\nu)}$ of this $\mathcal{H}$-action through $(\mu, \nu) \in \eta^*$

are that (1) $\mathcal{O}_{(\mu,0)}=\{(\mu,0)\}$, and (2) $\mathcal{O}_{(\mu,\nu\neq 0)}\cong \mathbb{R}^2\times\{\nu\}$. which are the immersed submanifolds of η^*.

We know that η^* is a Poisson manifold with respect to the $(\pm)$ magnetic Lie-Poisson bracket $\{\cdot,\cdot\}^B_{\pm}$ defined by

$$\{f,g\}^B_{\pm}(\mu,\nu):=\pm<(\mu,\nu),[\frac{\delta f}{\delta(\mu,\nu)},\frac{\delta g}{\delta(\mu,\nu)}]>-\pi^*_{\mathcal{H}}B(0,0)(\frac{\delta f}{\delta(\mu,\nu)},\frac{\delta g}{\delta(\mu,\nu)}), \tag{2.9}$$

for any $f,g\in C^\infty(\eta^*)$, and $(\mu,\nu)\in\eta^*$, where the element $\frac{\delta f}{\delta(\mu,\nu)}\in\eta$ is defined by the equality

$$<(\rho,\tau),\frac{\delta f}{\delta(\mu,\nu)}>:=Df(\mu,\nu)\cdot(\rho,\tau),$$

for any $(\rho,\tau)\in\eta^*$, see Marsden and Ratiu [9]. Thus, for the coadjoint orbit $\mathcal{O}_{(\mu,\nu)},\ (\mu,\nu)\in\eta^*$, the magnetic orbit symplectic structure can be defined by

$$\omega^{\pm}_{\mathcal{O}_{(\mu,\nu)}}(\rho,\tau)(\mathrm{ad}^*_{(X,a)}(\rho,\tau),\mathrm{ad}^*_{(Y,b)}(\rho,\tau))=\pm\langle(\rho,\tau),[(X,a),(Y,b)]\rangle-\pi^*_{\mathcal{H}}B(0,0)((X,a),(Y,b)), \tag{2.10}$$

for any $(X,a),\ (Y,b)\in\eta$, and $(\rho,\tau)\in\mathcal{O}_{(\mu,\nu)}\subset\eta^*$, which are coincide with the restriction of the magnetic Lie-Poisson brackets on η^* to the coadjoint orbit $\mathcal{O}_{(\mu,\nu)}$. From the Symplectic Stratification theorem we know that a finite dimensional Poisson manifold is the disjoint union of its symplectic leaves, and its each symplectic leaf is an injective immersed Poisson submanifold whose induced Poisson structure is symplectic. In consequence, when η^* is endowed one of the magnetic Lie-Poisson structures $\{\cdot,\cdot\}^B_{\pm}$, the symplectic leaves of the Poisson manifolds $(\eta^*,\{\cdot,\cdot\}^B_{\pm})$ coincide with the connected components of the magnetic orbits of the elements in η^* under the coadjoint action.

3 Magnetic Cotangent Bundle Reduction

In this section, we consider the magnetic reduction of cotangent bundle of the Heisenberg group $T^*\mathcal{H}$ with magnetic symplectic form $\omega_B=\omega_0-\pi^*_{\mathcal{H}}B$, where ω_0 is the usual canonical symplectic form on $T^*\mathcal{H}$, and B is a closed two-form on $\mathcal{H}$, and $\pi^*_{\mathcal{H}}B$ is the magnetic term on $T^*\mathcal{H}$, the map $\pi_{\mathcal{H}}:T^*\mathcal{H}\to\mathcal{H}$ is the cotangent bundle projection and $\pi^*_{\mathcal{H}}:T^*\mathcal{H}\to T^*T^*\mathcal{H}$. Defined the left $\mathcal{H}$-action $\Phi:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ given by

$$\Phi((u,\alpha),(v,\beta)):=(u,\alpha)(v,\beta)=(u+v,\alpha+\beta+\frac{1}{2}\omega(u,v)), \tag{3.1}$$

for any $(u,\alpha),(v,\beta)\in\mathcal{H}$, that is, the $\mathcal{H}$-action on $\mathcal{H}$ is the left translation on $\mathcal{H}$, which is free and proper, and leaves the two-form B invariant. By using the left trivialization of $T^*\mathcal{H}$, we have that $T^*\mathcal{H}=\mathcal{H}\times\eta^*$. We consider the cotangent lift of the $\mathcal{H}$-action to the magnetic cotangent bundle $(T^*\mathcal{H},\omega_B)$, which is given by

$$\Phi^{T^*}:\mathcal{H}\times T^*\mathcal{H}\to T^*\mathcal{H},\ \Phi^{T^*}((u,\alpha),((v,\beta),(\mu,\nu))):=((u,\alpha)(v,\beta),(\mu,\nu)), \tag{3.2}$$

for any $(u,\alpha),(v,\beta)\in\mathcal{H},\ (\mu,\nu)\in\eta^*$, and it is also a free and proper action.

If the $\mathcal{H}$-action on $(T^*\mathcal{H}, \omega_B)$ is symplectic, and admits an Ad^*-equivariant momentum map $\mathbf{J}: T^*\mathcal{H} \to \eta^*$. Assume that $(\mu,\nu) \in \eta^*$ is a regular value of $\mathbf{J}$ and $G_{(\mu,\nu)} = \{(u,\alpha) \in \mathcal{H} | \mathrm{Ad}^*_{(u,\alpha)}(\mu,\nu) = (\mu,\nu)\}$ is the isotropy subgroup of coadjoint $\mathcal{H}$-action at the point (μ,ν). Since $G_{(\mu,\nu)}(\subset \mathcal{H})$ acts freely and properly on $\mathcal{H}$ and on $T^*\mathcal{H}$, it follows that $G_{(\mu,\nu)}$ acts also freely and properly on $\mathbf{J}^{-1}((\mu,\nu))$, so that the space $(T^*\mathcal{H})_{(\mu,\nu)} = \mathbf{J}^{-1}((\mu,\nu))/G_{(\mu,\nu)}$ is a symplectic manifold with the reduced symplectic form $\omega_{(\mu,\nu)}$ uniquely characterized by the relation

$$\pi^*_{(\mu,\nu)}\omega_{(\mu,\nu)} = i^*_{(\mu,\nu)}\omega_B. \tag{3.3}$$

The map $i_{(\mu,\nu)}: \mathbf{J}^{-1}((\mu,\nu)) \to T^*\mathcal{H}$ is the inclusion and $\pi_{(\mu,\nu)}: \mathbf{J}^{-1}((\mu,\nu)) \to (T^*\mathcal{H})_{(\mu,\nu)}$ is the projection. The pair $((T^*\mathcal{H})_{(\mu,\nu)}, \omega_{(\mu,\nu)})$ is called the magnetic symplectic point reduced space of $(T^*\mathcal{H}, \omega_B)$ at (μ,ν).

If $(T^*\mathcal{H}, \omega_B)$ is a connected magnetic symplectic manifold, and $\mathbf{J}: T^*\mathcal{H} \to \eta^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma: \mathcal{H} \to \eta^*$, which is defined by $\sigma((u,\alpha)) := \mathbf{J}((u,\alpha)\cdot z) - \mathrm{Ad}^*_{(u,\alpha)^{-1}}\mathbf{J}(z)$, where $(u,\alpha) \in \mathcal{H}$ and $z \in T^*\mathcal{H}$. Then we know that σ produces a new affine action $\Theta: \mathcal{H} \times \eta^* \to \eta^*$ defined by

$$\Theta((u,\alpha),(\mu,\nu)) := \mathrm{Ad}^*_{(u,\alpha)^{-1}}(\mu,\nu) + \sigma((u,\alpha)), \tag{3.4}$$

where $(u,\alpha) \in \mathcal{H},\ (\mu,\nu) \in \eta^*$, with respect to which the given momentum map $\mathbf{J}$ is equivariant. Since $\mathcal{H}$ acts freely and properly on $T^*\mathcal{H}$, and $\tilde{G}_{(\mu,\nu)}$ denotes the isotropy subgroup of $(\mu,\nu) \in \eta^*$ relative to this affine action Θ and (μ,ν) is a regular value of $\mathbf{J}$. Then the quotient space $(T^*\mathcal{H})_{(\mu,\nu)} = \mathbf{J}^{-1}((\mu,\nu))/\tilde{G}_{(\mu,\nu)}$ is also a symplectic manifold with the reduced symplectic form $\omega_{(\mu,\nu)}$ uniquely characterized by (3.1), see Ortega and Ratiu [13].

Moreover, from Abraham and Marsden [1] and the discussion in §2, we can obtain the following theorem, which states that we can describe the magnetic symplectic point reduced space by using the coadjoint orbit with the magnetic orbit symplectic structure.

Theorem 3.1 *The coadjoint orbit $(\mathcal{O}_{(\mu,\nu)}, \omega^-_{\mathcal{O}_{(\mu,\nu)}})$, $(\mu,\nu) \in \eta^*$, is symplectically diffeomorphic to the magnetic symplectic point reduced space $((T^*\mathcal{H})_{(\mu,\nu)}, \omega_{(\mu,\nu)})$ of the magnetic cotangent bundle $(T^*\mathcal{H}, \omega_B)$.*

In the following we shall state that the magnetic term is related to a curvature two-form of a mechanical connection, by the reduction of center action of the Heisenberg group $\mathcal{H}$. In §2, we have known that the center of $\mathcal{H}$ is the subgroup $A = \{(0,\alpha) \in \mathcal{H} | \alpha \in \mathbb{R}\} \cong \mathbb{R}$, and the Heisenberg group is the central extension of $\mathbb{R}^2$ by $\mathbb{R}$, hence we can consider $\mathcal{H}$ as a right principal $\mathbb{R}$-bundle $\mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R} \to \mathbb{R}^2$, and there is an induced right $\mathcal{H}$-invariant metric on $\mathcal{H}$ as follows. In fact, we define that $<,>: \eta \times \eta \to \mathbb{R}$ given by $<(X,a),(Y,b)>= (X,Y) + ab$, for any $(X,a),\ (Y,b) \in \eta$, where the Euclidean inner product $(,)$ in $\mathbb{R}^2$ is used in the first summand and the multiplication of real numbers in the second summand. For $(u,\alpha) \in \mathcal{H}$ and $(X,a) \in T_{(u,\alpha)}\mathcal{H}$, the tangent of right translation on $\mathcal{H}$ is given by

$$T_{(u,\alpha)}R_{(v,\beta)}(X,a) = (X, a + \frac{1}{2}\omega(X,v)) \in T_{(u,\alpha)(v,\beta)}\mathcal{H}, \tag{3.5}$$

and, in particular, we have that

$$T_{(u,\alpha)}R_{(u,\alpha)^{-1}}(X,a) = (X, a - \frac{1}{2}\omega(X,u)) \in \eta. \tag{3.6}$$

Thus, the associated right $\mathcal{H}$-invariant metric on $\mathcal{H}$ is given by

$$\ll (X,a),(Y,b) \gg_{(u,\alpha)} = (X,Y) + ab - \frac{1}{2}a\omega(Y,u) - \frac{1}{2}b\omega(X,u) + \frac{1}{4}\omega(X,u)\omega(Y,u). \quad (3.7)$$

Note that the exponential map of the Heisenberg group $exp : \eta \to \mathcal{H}$ coincides with that of the vector Lie group $(\mathbb{R}^3, +)$, that is, the identity map of $\mathbb{R}^3$. For a given $a \in \mathbb{R}$, the infinitesimal generator for the right $\mathbb{R}$-action of $\mathcal{H}$ on $\mathcal{H}$ is given by

$$a_{\mathcal{H}}(v,\beta) = \frac{d}{dt}(v,\beta)(0,ta) = (0,a).$$

By combining these formulas and using the general formula for the locked inertia tensor, see Marsden [5], we can get the expression of the associated locked inertia tensor, that is,

$$< \mathbb{I}_{(u,\alpha)}(a), b > = \ll a_{\mathcal{H}}(u,\alpha), b_{\mathcal{H}}(u,\alpha) \gg_{(u,\alpha)} = ab, \quad (3.8)$$

for any $a,\ b \in \mathbb{R}$. Moreover, for any $(X,a) \in T_{(u,\alpha)}\mathcal{H}$ and $b \in \mathbb{R}$, we have the momentum map for the $\mathbb{R}$-action on $\mathcal{H}$ given by

$$< \mathbf{J}_{\mathbb{R}}(\ll (X,a), \cdot \gg_{(u,\alpha)}), b > = \ll (X,a),(0,b) \gg_{(u,\alpha)} = (a - \frac{1}{2}\omega(X,u))b. \quad (3.9)$$

Thus, we can get the expression of the associated mechanical connection $\mathcal{A} : T\mathcal{H} \to \mathbb{R}$ given by

$$\mathcal{A}(u,\alpha)(X,a) = a - \frac{1}{2}\omega(X,u), \quad (3.10)$$

and its exterior derivative is given by

$$\mathcal{B} = \mathbf{d}\mathcal{A}(u,\alpha)((X,a),(Y,b)) = \omega(X,Y), \quad (3.11)$$

where $(X,a),\ (Y,b) \in T_{(u,\alpha)}\mathcal{H}$, which offers a Lie algebra valued, closed two-form $\mathcal{B}$ on $\mathcal{H}$, that is, the curvature two-form of the mechanical connection $\mathcal{A}$. For $\nu \in \mathbb{R}^* \cong \mathbb{R}$, we can define the ν-component of $\mathcal{B}$ by $B^\nu = \nu\mathcal{B}$, such that for any $(u,\alpha) \in \mathcal{H},\ (X,a),\ (Y,b) \in T_{(u,\alpha)}\mathcal{H},\ B^\nu(u,\alpha)((X,a),(Y,b)) = \nu\mathcal{B}(u,\alpha)((X,a),(Y,b)) = \nu\omega(X,Y)$. This B^ν is an ordinary closed two-form on $\mathcal{H}$, and $\pi^*_{\mathcal{H}}B^\nu$ is usually the magnetic term on $T^*\mathcal{H}$. To sum up the above discussion, we have the following theorem.

Theorem 3.2 *There is a magnetic term on the cotangent bundle of the Heisenberg group $\mathcal{H}$, which is related to a curvature two-form of a mechanical connection determined by the reduction of center action of the Heisenberg group $\mathcal{H}$.*

4 RCH System and Its Equivalence

In order to describe the magnetic reduction of a regular controlled Hamiltonian (RCH) system with symmetry of the Heisenberg group $\mathcal{H}$ and the magnetic reducible controlled Hamiltonian (MRCH) equivalence, in this section, we shall review some relevant definitions and basic facts about RCH system, RCH-equivalence, regular point reducible RCH system and RpCH-equivalence. See Marsden et al. [10] for more details. Since the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle T^*Q may not be a Hamiltonian system on a cotangent bundle, that is, the set of Hamiltonian systems on the cotangent

bundle is not complete under the regular reduction, then in order to describe uniformly RCH systems defined on a cotangent bundle and on the regular reduced spaces, in the following we first define a RCH system on a symplectic fiber bundle. Thus, we can obtain the RCH system on the cotangent bundle of a configuration manifold as a special case, and discuss RCH-equivalence. For convenience, we assume that all controls appearing in this paper are the admissible controls.

Let (E, M, N, π, G) be a fiber bundle and (E, ω_E) be a symplectic fiber bundle. If for any function $H : E \to \mathbb{R}$, we have a Hamiltonian vector field X_H by $i_{X_H}\omega_E = \mathbf{d}H$, then (E, ω_E, H) is a Hamiltonian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle E as follows.

Definition 4.1 *(RCH System) A RCH system on E is a 5-tuple $(E, \omega_E, H, F, \mathcal{C})$, where (E, ω_E, H) is a Hamiltonian system, and the function $H : E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F : E \to E$ is called the (external) force map, and a fiber sub-manifold $\mathcal{C}$ of E is called the control subset.*

Sometimes, $\mathcal{C}$ also denotes the set of fiber-preserving maps from E to $\mathcal{C}$. When a feedback control law $u : E \to \mathcal{C}$ is chosen, the 5-tuple (E, ω_E, H, F, u) denotes a closed-loop dynamic system. In particular, when Q is a smooth manifold, and T^*Q its cotangent bundle with a symplectic form ω (not necessarily canonical symplectic form), then (T^*Q, ω) is a symplectic vector bundle. If we take that $E = T^*Q$, from above definition we can obtain a RCH system on the cotangent bundle T^*Q, that is, 5-tuple $(T^*Q, \omega, H, F, \mathcal{C})$. Here the fiber-preserving map $F : T^*Q \to T^*Q$ is the (external) force map, which is the reason that the fiber-preserving map $F : E \to E$ is called an (external) force map in above definition.

In order to describe the dynamics of the RCH system $(E, \omega_E, H, F, \mathcal{C})$ with a control law u, we can give a good expression of the dynamical vector field of RCH system by using the notations of vertical lifted maps of a vector along a fiber, see Marsden et al. [10]. In particular, in the case of cotangent bundle, for a given RCH System $(T^*Q, \omega, H, F, \mathcal{C})$, the dynamical vector field of the associated Hamiltonian system (T^*Q, ω, H) is that $X_H = (\mathbf{d}H)^\sharp$, where, $\sharp : T^*T^*Q \to TT^*Q;\ \mathbf{d}H \mapsto (\mathbf{d}H)^\sharp$, such that $i_{(\mathbf{d}H)^\sharp}\omega = \mathbf{d}H$. If considering the external force $F : T^*Q \to T^*Q$, by using the notation of vertical lifted map of a vector along a fiber, the change of X_H under the action of F is that

$$\mathrm{vlift}(F)X_H(\alpha_x) = \mathrm{vlift}((TFX_H)(F(\alpha_x)), \alpha_x) = (TFX_H)^v_\gamma(\alpha_x),$$

where $\alpha_x \in T^*_xQ,\ x \in Q$ and γ is a straight line in T^*_xQ connecting $F_x(\alpha_x)$ and α_x. In the same way, when a feedback control law $u : T^*Q \to \mathcal{C}$ is chosen, the change of X_H under the action of u is that

$$\mathrm{vlift}(u)X_H(\alpha_x) = \mathrm{vlift}((TuX_H)(u(\alpha_x)), \alpha_x) = (TuX_H)^v_\gamma(\alpha_x).$$

In consequence, we can give an expression of the dynamical vector field of RCH system as follows.

Proposition 4.2 *The dynamical vector field of a RCH system $(T^*Q, \omega, H, F, \mathcal{C})$ with a control law u is the synthetic of Hamiltonian vector field X_H and its changes under the actions of the external force F and control u, that is,*

$$X_{(T^*Q,\omega,H,F,u)}(\alpha_x) = X_H(\alpha_x) + \mathrm{vlift}(F)X_H(\alpha_x) + \mathrm{vlift}(u)X_H(\alpha_x),$$

for any $\alpha_x \in T^*_xQ,\ x \in Q$. For convenience, it is simply written as

$$X_{(T^*Q,\omega,H,F,u)} = X_H + \mathrm{vlift}(F) + \mathrm{vlift}(u). \tag{4.1}$$

We also denote that $\mathrm{vlift}(\mathcal{C}) = \bigcup\{\mathrm{vlift}(u)X_H|\ u \in \mathcal{C}\}$. For the RCH system $(E,\omega_E,H,F,\mathcal{C})$ with a control law u, we have also a similar expression of its dynamical vector field. It is worthy of note that in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q,\omega,H,F,u)}$.

Next, we know that when a RCH system is given, the force map F is determined, but the feedback control law $u : T^*Q \to \mathcal{C}$ could be chosen. In order to describe the feedback control law to modify the structure of RCH system, the controlled Hamiltonian matching conditions and RCH-equivalence are induced as follows.

Definition 4.3 *(RCH-equivalence) Suppose that we have two RCH systems $(T^*Q_i,\omega_i,H_i,F_i,\mathcal{C}_i)$, $i = 1,2$, we say them to be RCH-equivalent, or simply, $(T^*Q_1,\omega_1,H_1,F_1,\mathcal{C}_1) \overset{RCH}{\sim}$ $(T^*Q_2,\omega_2,H_2,F_2,\mathcal{C}_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that the following controlled Hamiltonian matching conditions hold:*

RCH-1: *The cotangent lifted map of φ, that is, $\varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1$ is symplectic, and $\mathcal{C}_1 = \varphi^*(\mathcal{C}_2)$.*

RCH-2: *$Im[X_{H_1} + \mathrm{vlift}(F_1) - T\varphi^* X_{H_2} - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(\mathcal{C}_1)$, where the map $\varphi_* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2$, and $T\varphi^* : TT^*Q_2 \to TT^*Q_1$, and Im means the pointwise image of the map in brackets.*

In the following we consider the RCH system with symmetry, and give the regular point reducible RCH system and RpCH-equivalence. Let $\Phi : G \times Q \to Q$ be a smooth left action of the Lie group G on Q, which is free and proper. Then the cotangent lifted left action $\Phi^{T^*} : G \times T^*Q \to T^*Q$ is symplectic, free and proper, and admits a Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra of G and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $\mathbf{J}$ and denote by G_μ the isotropy subgroup of the coadjoint G-action at the point $\mu \in \mathfrak{g}^*$, which is defined by $G_\mu = \{g \in G|\ \mathrm{Ad}^*_g \mu = \mu\}$. Since $G_\mu (\subset G)$ acts freely and properly on T^*Q, then G_μ acts also freely and properly on $\mathbf{J}^{-1}(\mu)$, so that the space $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation

$$\pi^*_\mu \omega_\mu = i^*_\mu \omega. \tag{4.2}$$

The map $i_\mu : \mathbf{J}^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. The pair $((T^*Q)_\mu, \omega_\mu)$ is called Marsden-Weinstein reduced space of (T^*Q,ω) at μ. On the other hand, if (T^*Q,ω) is a connected symplectic manifold, and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to \mathfrak{g}^*$, which is defined by $\sigma(g) := \mathbf{J}(g\cdot z) - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that σ produces a new affine action $\Theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Theta(g,\mu) := \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g)$, where $\mu \in \mathfrak{g}^*$, with respect to which the given momentum map $\mathbf{J}$ is equivariant. Assume that G acts freely and properly on T^*Q, and μ is a regular value of $\mathbf{J}$, and $\tilde{G_\mu}$ denotes the isotropy subgroup of $\mu \in \mathfrak{g}^*$ relative to this affine action Θ. Then the quotient space $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/\tilde{G_\mu}$ is also symplectic manifold. See Marsden et al. [10] for more details.

Assume that $H : T^*Q \to \mathbb{R}$ is a G-invariant Hamiltonian, the flow F_t of the Hamiltonian vector field X_H leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the

G-action, so it induces a flow f_t^μ on $(T^*Q)_\mu$, defined by $f_t^\mu \cdot \pi_\mu = \pi_\mu \cdot F_t \cdot i_\mu$, and the vector field X_{h_μ} generated by the flow f_t^μ on $((T^*Q)_\mu, \omega_\mu)$ is Hamiltonian with the associated regular point reduced Hamiltonian function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ defined by $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, and the Hamiltonian vector fields X_H and X_{h_μ} are π_μ-related. On the other hand, from Marsden et al. [10], we know that the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a regular point reducible RCH system as follows.

Definition 4.4 *(Regular Point Reducible RCH System) A 6-tuple $(T^*Q, G, \omega, H, F, \mathcal{C})$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold $\mathcal{C}$ of T^*Q are all G-invariant, is called a regular point reducible RCH system, if there exists a point $\mu \in \mathfrak{g}^*$, which is a regular value of the momentum map $\mathbf{J}$, such that the regular point reduced system, that is, the 5-tuple $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, \mathcal{C}_\mu)$, where $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$, $\mathcal{C} \subset \mathbf{J}^{-1}(\mu)$, $\mathcal{C}_\mu = \pi_\mu(\mathcal{C})$, is a RCH system, which is simply written as R_P-reduced RCH system. Where $((T^*Q)_\mu, \omega_\mu)$ is the R_P-reduced space, which is also called Marsden-Weinstein reduced space, the function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ is called the reduced Hamiltonian, the fiber-preserving map $f_\mu : (T^*Q)_\mu \to (T^*Q)_\mu$ is called the reduced (external) force map, $\mathcal{C}_\mu$ is a fiber sub-manifold of $(T^*Q)_\mu$ and is called the reduced control subset.*

Denote by $X_{(T^*Q,G,\omega,H,F,u)}$ the vector field of the regular point reducible RCH system $(T^*Q, G, \omega, H, F, \mathcal{C})$ with a control law u, then

$$X_{(T^*Q,G,\omega,H,F,u)} = X_H + \mathrm{vlift}(F) + \mathrm{vlift}(u). \tag{4.3}$$

Moreover, for the regular point reducible RCH system we can also introduce the regular point reduced controlled Hamiltonian equivalence (RpCH-equivalence) as follows.

Definition 4.5 *(RpCH-equivalence) Suppose that we have two regular point reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, we say them to be RpCH-equivalent, or simply, $(T^*Q_1, G_1, \omega_1, H_1, F_1, \mathcal{C}_1) \overset{RpCH}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, \mathcal{C}_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following controlled Hamiltonian matching conditions hold:*
RpCH-1: *The cotangent lifted map $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic.*
RpCH-2: *For $\mu_i \in \mathfrak{g}_i^*$, the regular reducible points of RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, the map $\varphi_\mu^* = i_{\mu_1}^{-1} \cdot \varphi^* \cdot i_{\mu_2} : \mathbf{J}_2^{-1}(\mu_2) \to \mathbf{J}_1^{-1}(\mu_1)$ is $(G_{2\mu_2}, G_{1\mu_1})$-equivariant and $\mathcal{C}_1 = \varphi_\mu^*(\mathcal{C}_2)$, where $\mu = (\mu_1, \mu_2)$, and denotes $i_{\mu_1}^{-1}(S)$ by the pre-image of a subset $S \subset T^*Q_1$ for the map $i_{\mu_1} : \mathbf{J}_1^{-1}(\mu_1) \to T^*Q_1$.*
RpCH-3: *$Im[X_{H_1} + \mathrm{vlift}(F_1) - T\varphi^* X_{H_2} - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(\mathcal{C}_1)$.*

It is worthy of note that for the regular point reducible RCH system, the induced equivalent map φ^* not only keeps the symplectic structure, but also keeps the equivariance of G-action at the regular point. If a feedback control law $u_\mu : (T^*Q)_\mu \to \mathcal{C}_\mu$ is chosen, the R_P-reduced RCH system $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)$ is a closed-loop regular dynamic system with a control law u_μ. Assume that its vector field $X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)}$ can be expressed by

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} = X_{h_\mu} + \mathrm{vlift}(f_\mu) + \mathrm{vlift}(u_\mu), \tag{4.4}$$

where X_{h_μ} is the vector field of the reduced Hamiltonian h_μ, $\mathrm{vlift}(f_\mu) = \mathrm{vlift}(f_\mu)X_{h_\mu}$, $\mathrm{vlift}(u_\mu) = \mathrm{vlift}(u_\mu)X_{h_\mu}$, and satisfies the condition

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} \cdot \pi_\mu = T\pi_\mu \cdot X_{(T^*Q,G,\omega,H,F,u)} \cdot i_\mu. \tag{4.5}$$

Then we can obtain the following regular point reduction theorem for the RCH system, which explains the relationship between the RpCH-equivalence for the regular point reducible RCH systems with symmetry and the RCH-equivalence for the associated R_P-reduced RCH systems, its proof is given in Marsden et al. [10]. This theorem can be regarded as an extension of the regular point reduction theorem of a Hamiltonian system under regular controlled Hamiltonian equivalence conditions.

Theorem 4.6 *Two regular point reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, are RpCH-equivalent if and only if the associated R_P-reduced RCH systems $((T^*Q_i)_{\mu_i}, \omega_{i\mu_i}, h_{i\mu_i}, f_{i\mu_i}, \mathcal{C}_{i\mu_i})$, $i = 1, 2$, are RCH-equivalent.*

5 Magnetic Reduction of RCH System and MRCH-equivalence

In this section, we shall consider the magnetic reduction of the regular controlled Hamiltonian system $(T^*Q, \mathcal{H}, \omega_Q, H, F, \mathcal{C})$ with symmetry of the Heisenberg group $\mathcal{H}$. Here the configuration space $Q = \mathcal{H} \times V$, $\mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R}$, and V is a k-dimensional vector space, and the cotangent bundle T^*Q with magnetic symplectic form $\omega_Q = \Omega_0 - \pi_Q^* \bar{B}$, where Ω_0 is the usual canonical symplectic form on T^*Q, and $\bar{B} = \pi_1^* B$ is the closed two-form on Q, B is a closed two-form on $\mathcal{H}$ and $\pi_1 : Q = \mathcal{H} \times V \to \mathcal{H}$ and $\pi_1^* : T^*\mathcal{H} \to T^*Q$. Defined the left $\mathcal{H}$-action Φ on Q as follows

$$\Phi : \mathcal{H} \times Q \to Q, \quad \Phi((u, \alpha), ((v, \beta), \theta)) := ((u, \alpha)(v, \beta), \theta), \tag{5.1}$$

for any $(u, \alpha), (v, \beta) \in \mathcal{H}$, $\theta \in V$, that is, the $\mathcal{H}$-action on Q is the left translation on the first factor $\mathcal{H}$, and $\mathcal{H}$ acts trivially on the second factor V. Because $T^*Q = T^*\mathcal{H} \times T^*V$, and $T^*V = V \times V^*$, by using the left trivialization of $T^*\mathcal{H}$, we have that $T^*Q = \mathcal{H} \times \eta^* \times V \times V^*$. If the left $\mathcal{H}$-action $\Phi : \mathcal{H} \times Q \to Q$ is free and proper, then the cotangent lift of the action to its cotangent bundle T^*Q is given by

$$\Phi^{T^*} : \mathcal{H} \times T^*Q \to T^*Q, \ \Phi^{T^*}((u, \alpha), ((v, \beta), (\mu, \nu), \theta, \lambda)) := ((u, \alpha)(v, \beta), (\mu, \nu), \theta, \lambda), \tag{5.2}$$

for any $(u, \alpha), (v, \beta) \in \mathcal{H}$, $(\mu, \nu) \in \eta^*$, $\theta \in V$, $\lambda \in V^*$, and it is also a free and proper action, and the orbit space $(T^*Q)/\mathcal{H}$ is a smooth manifold and $\pi : T^*Q \to (T^*Q)/\mathcal{H}$ is a smooth submersion. Since $\mathcal{H}$ acts trivially on η^*, V and V^*, it follows that $(T^*Q)/\mathcal{H}$ is diffeomorphic to $\eta^* \times V \times V^*$.

From §2 we have know that for $(\mu, \nu) \in \eta^*$, the coadjoint orbit $\mathcal{O}_{(\mu,\nu)} \subset \eta^*$ has the magnetic orbit symplectic forms $\omega^{\pm}_{\mathcal{O}_{(\mu,\nu)}}$ given by (2.10). Let ω_V be the canonical symplectic form on $T^*V \cong V \times V^*$ given by

$$\omega_V((\theta_1, \lambda_1), (\theta_2, \lambda_2)) =< \lambda_2, \theta_1 > - < \lambda_1, \theta_2 >,$$

where $(\theta_i, \lambda_i) \in V \times V^*$, $i = 1, 2$, $< \cdot, \cdot >$ is the natural pairing between V^* and V. Thus, we can induce a symplectic forms $\tilde{\omega}^{\pm}_{\tilde{\mathcal{O}}_{(\mu,\nu)}} = \pi^*_{\mathcal{O}_{(\mu,\nu)}} \omega^{\pm}_{\mathcal{O}_{(\mu,\nu)}} + \pi_V^* \omega_V$ on the smooth manifold $\tilde{\mathcal{O}}_{(\mu,\nu)} = \mathcal{O}_{(\mu,\nu)} \times V \times V^*$, where the maps $\pi_{\mathcal{O}_{(\mu,\nu)}} : \mathcal{O}_{(\mu,\nu)} \times V \times V^* \to \mathcal{O}_{(\mu,\nu)}$ and $\pi_V : \mathcal{O}_{(\mu,\nu)} \times V \times V^* \to V \times V^*$ are canonical projections.

On the other hand, from $T^*Q = T^*\mathcal{H} \times T^*V$ we know that there is a canonical symplectic form $\Omega_0 = \pi_1^* \omega_0 + \pi_2^* \omega_V$ on T^*Q, where ω_0 is the canonical symplectic form on $T^*\mathcal{H}$ and the maps $\pi_1 : Q = \mathcal{H} \times V \to \mathcal{H}$ and $\pi_2 : Q = \mathcal{H} \times V \to V$ are canonical projections. Then the magnetic

symplectic form on T^*Q is given by $\omega_Q = \pi_1^*\omega_0 + \pi_2^*\omega_V - \pi_Q^* \cdot \pi_1^* B$. If the cotangent lift of left $\mathcal{H}$-action $\Phi^{T^*} : \mathcal{H} \times T^*Q \to T^*Q$ is symplectic with respect to ω_Q, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q : T^*Q \to \eta^*$ such that $\mathbf{J}_Q \cdot \pi_1^* = \mathbf{J}_{\mathcal{H}}$, where $\mathbf{J}_{\mathcal{H}} : T^*\mathcal{H} \to \eta^*$ is a momentum map of left $\mathcal{H}$-action on $T^*\mathcal{H}$, and $\pi_1^* : T^*\mathcal{H} \to T^*Q$. If $(\mu,\nu) \in \eta^*$ is a regular value of $\mathbf{J}_Q$, then $(\mu,\nu) \in \eta^*$ is also a regular value of $\mathbf{J}_{\mathcal{H}}$ and $\mathbf{J}_Q^{-1}((\mu,\nu)) \cong \mathbf{J}_{\mathcal{H}}^{-1}((\mu,\nu)) \times V \times V^*$. Denote by $G_{(\mu,\nu)} = \{(u,\alpha) \in \mathcal{H} | \, \mathrm{Ad}^*_{(u,\alpha)}(\mu,\nu) = (\mu,\nu)\}$ the isotropy subgroup of coadjoint $\mathcal{H}$-action at the point $(\mu,\nu) \in \eta^*$. It follows that $G_{(\mu,\nu)}$ acts also freely and properly on $\mathbf{J}_Q^{-1}((\mu,\nu))$, the magnetic point reduced space

$$(T^*Q)_{(\mu,\nu)} = \mathbf{J}_Q^{-1}((\mu,\nu))/G_{(\mu,\nu)} \cong (T^*\mathcal{H})_{(\mu,\nu)} \times V \times V^*$$

of (T^*Q, ω_Q) at (μ,ν), is a symplectic manifold with the reduced symplectic form $\omega_{(\mu,\nu)}$ uniquely characterized by the relation

$$\pi^*_{(\mu,\nu)}\omega_{(\mu,\nu)} = i^*_{(\mu,\nu)}\omega_Q = i^*_{(\mu,\nu)}\pi_1^*\omega_0 + i^*_{(\mu,\nu)}\pi_2^*\omega_V - i^*_{(\mu,\nu)}\pi_Q^* \cdot \pi_1^* B, \tag{5.3}$$

where the map $i_{(\mu,\nu)} : \mathbf{J}_Q^{-1}((\mu,\nu)) \to T^*Q$ is the inclusion and $\pi_{(\mu,\nu)} : \mathbf{J}_Q^{-1}((\mu,\nu)) \to (T^*Q)_{(\mu,\nu)}$ is the projection. From Theorem 3.1 we have seen that $((T^*\mathcal{H})_{(\mu,\nu)}, \omega_{(\mu,\nu)})$ is symplectically diffeomorphic to $(\mathcal{O}_{(\mu,\nu)}, \omega^-_{\mathcal{O}_{(\mu,\nu)}})$, hence we obtain that $((T^*Q)_{(\mu,\nu)}, \omega_{(\mu,\nu)})$ is symplectically diffeomorphic to $(\tilde{\mathcal{O}}_{(\mu,\nu)} = \mathcal{O}_{(\mu,\nu)} \times V \times V^*, \ \tilde{\omega}^-_{\tilde{\mathcal{O}}_{(\mu,\nu)}})$.

Remark 5.1 *If (T^*Q, ω_Q) is a connected magnetic symplectic manifold, and $\mathbf{J}_Q : T^*Q \to \eta^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : \mathcal{H} \to \eta^*$, which is defined by $\sigma((u,\alpha)) := \mathbf{J}_Q((u,\alpha) \cdot z) - \mathrm{Ad}^*_{(u,\alpha)^{-1}} \mathbf{J}_Q(z)$, where $(u,\alpha) \in \mathcal{H}$ and $z \in T^*Q$. Then we know that σ produces a new affine action $\Theta : \mathcal{H} \times \eta^* \to \eta^*$ defined by*

$$\Theta((u,\alpha),(\mu,\nu)) := \mathrm{Ad}^*_{(u,\alpha)^{-1}}(\mu,\nu) + \sigma((u,\alpha)), \tag{5.4}$$

where $(u,\alpha) \in \mathcal{H}$, $(\mu,\nu) \in \eta^*$, with respect to which the given momentum map $\mathbf{J}_Q$ is equivariant. Since $\mathcal{H}$ acts freely and properly on T^*Q, and $\tilde{G}_{(\mu,\nu)}$ denotes the isotropy subgroup of $(\mu,\nu) \in \eta^*$ relative to this affine action Θ and (μ,ν) is a regular value of $\mathbf{J}_Q$. Then the magnetic point reduced space $(T^*Q)_{(\mu,\nu)} = \mathbf{J}_Q^{-1}((\mu,\nu))/\tilde{G}_{(\mu,\nu)}$ is also a symplectic manifold with the reduced symplectic form $\omega_{(\mu,\nu)}$ uniquely characterized by (5.3), and this space is symplectically diffeomorphic to $(\tilde{\mathcal{O}}_{(\mu,\nu)} = \mathcal{O}_{(\mu,\nu)} \times V \times V^*, \ \tilde{\omega}^-_{\tilde{\mathcal{O}}_{(\mu,\nu)}})$, where $\mathcal{O}_{(\mu,\nu)}$ is the coadjoint orbit at $(\mu,\nu) \in \eta^*$.

Now assume that Hamiltonian $H((u,\alpha),(\rho,\tau),\theta,\lambda) : T^*Q \cong \mathcal{H} \times \eta^* \times V \times V^* \to \mathbb{R}$ is left cotangent lifted $\mathcal{H}$-action Φ^{T^*} invariant, for the regular value $(\mu,\nu) \in \eta^*$ of $\mathbf{J}_Q$, we have the associated reduced Hamiltonian $h_{(\mu,\nu)}((\rho,\tau),\theta,\lambda) : (T^*Q)_{(\mu,\nu)} \cong \mathcal{O}_{(\mu,\nu)} \times V \times V^* \to \mathbb{R}$, defined by $h_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = H \cdot i_{(\mu,\nu)}$, and the reduced Hamiltonian vector field $X_{h_{(\mu,\nu)}}$ is given by the reduced Hamilton's equation $\mathbf{i}_{X_{h_{(\mu,\nu)}}}\omega_{(\mu,\nu)} = \mathbf{d}h_{(\mu,\nu)}$. Thus, if the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold $\mathcal{C}$ of T^*Q are all left cotangent lifted $\mathcal{H}$-action Φ^{T^*} invariant, then the 6-tuple $(T^*Q, \mathcal{H}, \omega_Q, H, F, \mathcal{C})$ is a magnetic point reducible RCH system. For the point $(\mu,\nu) \in \eta^*$, the regular value of the momentum map $\mathbf{J}_Q : T^*Q \to \eta^*$, and the given feedback control $u : T^*Q \to \mathcal{C}(\subset \mathbf{J}_Q^{-1}(\mu,\nu))$, the magnetic point reduced RCH system is the 5-tuple $(\mathcal{O}_{(\mu,\nu)} \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}, h_{(\mu,\nu)}, f_{(\mu,\nu)}, u_{(\mu,\nu)})$, where $\mathcal{O}_{(\mu,\nu)} \subset \eta^*$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}$ is the magnetic orbit symplectic form on $\mathcal{O}_{(\mu,\nu)} \times V \times V^*$, $h_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = H \cdot i_{(\mu,\nu)}$,

$f_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = \pi_{(\mu,\nu)} \cdot F \cdot i_{(\mu,\nu)}$, $\mathcal{C} \subset \mathbf{J}_Q^{-1}(\mu,\nu)$, and $u_{(\mu,\nu)} \in \mathcal{C}_{(\mu,\nu)} = \pi_{(\mu,\nu)}(\mathcal{C}) \subset \mathcal{O}_{(\mu,\nu)} \times V \times V^*$, $u_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = \pi_{(\mu,\nu)} \cdot u \cdot i_{(\mu,\nu)}$. Moreover, the dynamical vector field of the magnetic reduced RCH system can be expressed by

$$X_{(\mathcal{O}_{(\mu,\nu)} \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}, h_{(\mu,\nu)}, f_{(\mu,\nu)}, u_{(\mu,\nu)})} = X_{h_{(\mu,\nu)}} + \mathrm{vlift}(f_{(\mu,\nu)}) + \mathrm{vlift}(u_{(\mu,\nu)}), \tag{5.5}$$

where $X_{h_{(\mu,\nu)}} \in T(\mathcal{O}_{(\mu,\nu)} \times V \times V^*)$ is Hamiltonian vector field of the magnetic reduced Hamiltonian $h_{(\mu,\nu)} : \mathcal{O}_{(\mu,\nu)} \times V \times V^* \to \mathbb{R}$, and $\mathrm{vlift}(f_{(\mu,\nu)}) = \mathrm{vlift}(f_{(\mu,\nu)}) X_{h_{(\mu,\nu)}} \in T(\mathcal{O}_{(\mu,\nu)} \times V \times V^*)$, $\mathrm{vlift}(u_{(\mu,\nu)}) = \mathrm{vlift}(u_{(\mu,\nu)}) X_{h_{(\mu,\nu)}} \in T(\mathcal{O}_{(\mu,\nu)} \times V \times V^*)$, and satisfies the condition

$$X_{(\mathcal{O}_{(\mu,\nu)} \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}, h_{(\mu,\nu)}, f_{(\mu,\nu)}, u_{(\mu,\nu)})} \cdot \pi_{(\mu,\nu)} = T\pi_{(\mu,\nu)} \cdot X_{(T^*Q, \mathcal{H}, \omega_Q, H, F, u)} \cdot i_{(\mu,\nu)}. \tag{5.6}$$

Note that $\mathrm{vlift}(u_{(\mu,\nu)}) X_{h_{(\mu,\nu)}}$ is the vertical lift of vector field $X_{h_{(\mu,\nu)}}$ under the action of $u_{(\mu,\nu)}$ along fibers, that is,

$$\begin{aligned} \mathrm{vlift}(u_{(\mu,\nu)}) X_{h_{(\mu,\nu)}}((\rho,\tau),\theta,\lambda) &= \mathrm{vlift}((Tu_{(\mu,\nu)} X_{h_{(\mu,\nu)}})(u_{(\mu,\nu)}((\rho,\tau),\theta,\lambda)), ((\rho,\tau),\theta,\lambda)) \\ &= (Tu_{(\mu,\nu)} X_{h_{(\mu,\nu)}})^v_{\tilde{\sigma}}((\rho,\tau),\theta,\lambda), \end{aligned}$$

where $(\mu,\nu),\ (\rho,\tau) \in \eta^*,\ \theta \in V,\ \lambda \in V^*$, and $\tilde{\sigma}$ is a geodesic in $\mathcal{O}_{(\mu,\nu)} \times V \times V^*$ connecting $u_{(\mu,\nu)}((\rho,\tau),\theta,\lambda)$ and $((\rho,\tau),\theta,\lambda)$, and $(Tu_{(\mu,\nu)} X_{h_{(\mu,\nu)}})^v_{\tilde{\sigma}}((\rho,\tau),\theta,\lambda)$ is the parallel displacement of vertical vector $(Tu_{(\mu,\nu)} X_{h_{(\mu,\nu)}})^v((\rho,\tau),\theta,\lambda)$ along the geodesic $\tilde{\sigma}$ from $u_{(\mu,\nu)}((\rho,\tau),\theta,\lambda)$ to $((\rho,\tau),\theta,\lambda)$, and $\mathrm{vlift}(f_{(\mu,\nu)}) X_{h_{(\mu,\nu)}}$ is defined in the similar manner, see Marsden et al. [10] and Proposition 4.2 in §4. Moreover, from Definition 4.4, we can get the following theorem.

Theorem 5.2 *The 6-tuple $(T^*Q, \mathcal{H}, \omega_Q, H, F, \mathcal{C})$ is a magnetic point reducible RCH system with symmetry of the Heisenberg group $\mathcal{H}$, where $Q = \mathcal{H} \times V$, and $\mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R}$ is the Heisenberg group with Lie algebra $\eta \cong \mathbb{R}^2 \oplus \mathbb{R}$ and its dual $\eta^* \cong \mathbb{R}^2 \oplus \mathbb{R}$, and V is a k-dimensional vector space, and the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold $\mathcal{C}$ of T^*Q are all left cotangent lifted $\mathcal{H}$-action Φ^{T^*} invariant. For the point $(\mu,\nu) \in \eta^*$, the regular value of the momentum map $\mathbf{J}_Q : T^*Q \to \eta^*$, and the given feedback control $u : T^*Q \to \mathcal{C} \subset \mathbf{J}_Q^{-1}(\mu,\nu)$, the magnetic point reduced RCH system is the 5-tuple $(\mathcal{O}_{(\mu,\nu)} \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}, h_{(\mu,\nu)}, f_{(\mu,\nu)}, u_{(\mu,\nu)})$, where $\mathcal{O}_{(\mu,\nu)} \subset \eta^*$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_{(\mu,\nu)} \times V \times V^*}$ is the magnetic orbit symplectic form on $\mathcal{O}_{(\mu,\nu)} \times V \times V^*$, $h_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = H \cdot i_{(\mu,\nu)}$, $f_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = \pi_{(\mu,\nu)} \cdot F \cdot i_{(\mu,\nu)}$, $\mathcal{C} \subset \mathbf{J}_Q^{-1}(\mu,\nu)$, and $u_{(\mu,\nu)} \in \mathcal{C}_{(\mu,\nu)} = \pi_{(\mu,\nu)}(\mathcal{C}) \subset \mathcal{O}_{(\mu,\nu)} \times V \times V^*$, $u_{(\mu,\nu)} \cdot \pi_{(\mu,\nu)} = \pi_{(\mu,\nu)} \cdot u \cdot i_{(\mu,\nu)}$, and the dynamical vector field of the magnetic reduced RCH system is given by* (5.5).

Next, in order to describe the feedback control law to modify the structure of the magnetic point reducible RCH system with symmetry of the Heisenberg group, from Definition 4.5 we can give the magnetic reduced controlled Hamiltonian matching conditions and MRCH-equivalence as follows.

Definition 5.3 *(MRCH-equivalence) Suppose that we have two magnetic point reducible RCH systems with symmetry of the Heisenberg group $(T^*Q_i, \mathcal{H}, \omega_{Q_i}, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, we say them to be MRCH-equivalent, or simply, $(T^*Q_1, \mathcal{H}, \omega_{Q_1}, H_1, F_1, \mathcal{C}_1) \overset{MRCH}{\sim} (T^*Q_2, \mathcal{H}, \omega_{Q_2}, H_2, F_2, \mathcal{C}_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following magnetic reduced controlled Hamiltonian matching conditions hold:*

MR-1: *The cotangent lifted map* $\varphi^*: T^*Q_2 \to T^*Q_1$ *is symplectic with respect to their magnetic symplectic forms.*
MR-2: *For* $(\mu_i, \nu_i) \in \eta_i^*,\ i = 1, 2$, *the magnetic reducible points of RCH systems with symmetry of the Heisenberg group* $(T^*Q_i, \mathcal{H}, \omega_{Q_i}, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, *the map* $\varphi^*_{(\mu,\nu)} = i^{-1}_{(\mu_1,\nu_1)} \cdot \varphi^* \cdot i_{(\mu_2,\nu_2)} : \mathbf{J}^{-1}_{Q_2}(\mu_2, \nu_2) \to \mathbf{J}^{-1}_{Q_1}(\mu_1, \nu_1)$ *is* $(G_{2(\mu_2,\nu_2)}, G_{1(\mu_1,\nu_1)})$*-equivariant and* $\mathcal{C}_1 = \varphi^*_{(\mu,\nu)}(\mathcal{C}_2)$, *where* $(\mu, \nu) = ((\mu_1, \mu_2), (\nu_1, \nu_2))$, *and denotes* $i^{-1}_{(\mu_1,\nu_1)}(S)$ *by the pre-image of a subset* $S \subset T^*Q_1$ *for the map* $i_{(\mu_1,\nu_1)} : \mathbf{J}^{-1}_{Q_1}(\mu_1, \nu_1) \to T^*Q_1$.
MR-3: $Im[X_{H_1} + \mathrm{vlift}(F_1) - T\varphi^* X_{H_2} - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(\mathcal{C}_1)$, *where* Im *means the pointwise image of the map in brackets.*

Then, by using the method given in Marsden et al. [10], we can prove the following magnetic point reduction theorem for the RCH system with symmetry of the Heisenberg group, which explains the relationship between MRCH-equivalence for the magnetic point reducible RCH systems with symmetry of the Heisenberg group and RCH-equivalence for the associated magnetic point reduced RCH systems.

Theorem 5.4 *Two magnetic point reducible RCH systems with symmetry of the Heisenberg group* $(T^*Q_i, \mathcal{H}, \omega_{Q_i}, H_i, F_i, \mathcal{C}_i)$, $i = 1, 2$, *are MRCH-equivalent if and only if the associated magnetic point reduced RCH systems* $(\mathcal{O}_{i(\mu_i,\nu_i)} \times V_i \times V_i^*, \tilde{\omega}^-_{\mathcal{O}_{i(\mu_i,\nu_i)} \times V_i \times V_i^*}, h_{i(\mu_i,\nu_i)}, f_{i(\mu_i,\nu_i)}, \mathcal{C}_{i(\mu_i,\nu_i)})$, $i = 1, 2$, *are RCH-equivalent.*

Proof: Assume that $(T^*Q_1, \mathcal{H}, \omega_{Q_1}, H_1, F_1, \mathcal{C}_1) \overset{MRCH}{\sim} (T^*Q_2, \mathcal{H}, \omega_{Q_2}, H_2, F_2, \mathcal{C}_2)$, then from Definition 5.3 we know that there exists a diffeomorphism $\varphi: Q_1 \to Q_2$ such that $\varphi^*: T^*Q_2 \to T^*Q_1$ is symplectic with respect to their magnetic symplectic forms, and for $(\mu_i, \nu_i) \in \eta_i^*,\ i = 1, 2$, the map $\varphi^*_{(\mu,\nu)} = i^{-1}_{(\mu_1,\nu_1)} \cdot \varphi^* \cdot i_{(\mu_2,\nu_2)} : \mathbf{J}^{-1}_{Q_2}(\mu_2, \nu_2) \to \mathbf{J}^{-1}_{Q_1}(\mu_1, \nu_1)$ is $(G_{2(\mu_2,\nu_2)}, G_{1(\mu_1,\nu_1)})$-equivariant, and $\mathcal{C}_1 = \varphi^*_{(\mu,\nu)}(\mathcal{C}_2)$ and MR-3 holds. From the following commutative Diagram-1:

$$
\begin{array}{ccccc}
T^*Q_2 & \xleftarrow{i_{(\mu_2,\nu_2)}} & \mathbf{J}^{-1}_{Q_2}(\mu_2,\nu_2) & \xrightarrow{\pi_{(\mu_2,\nu_2)}} & (T^*Q_2)_{(\mu_2,\nu_2)} \\
{\scriptstyle \varphi^*}\downarrow & & {\scriptstyle \varphi^*_{(\mu,\nu)}}\downarrow & & {\scriptstyle \varphi^*_{(\mu,\nu)/\mathcal{H}}}\downarrow \\
T^*Q_1 & \xleftarrow{i_{(\mu_1,\nu_1)}} & \mathbf{J}^{-1}_{Q_1}(\mu_1,\nu_1) & \xrightarrow{\pi_{(\mu_1,\nu_1)}} & (T^*Q_1)_{(\mu_1,\nu_1)}
\end{array}
$$

Diagram-1

We can define a map $\varphi^*_{(\mu,\nu)/\mathcal{H}} : (T^*Q_2)_{(\mu_2,\nu_2)} \to (T^*Q_1)_{(\mu_1,\nu_1)}$ such that $\varphi^*_{(\mu,\nu)/\mathcal{H}} \cdot \pi_{(\mu_2,\nu_2)} = \pi_{(\mu_1,\nu_1)} \cdot \varphi^*_{(\mu,\nu)}$. Because $\varphi^*_{(\mu,\nu)} : \mathbf{J}^{-1}_{Q_2}(\mu_2, \nu_2) \to \mathbf{J}^{-1}_{Q_1}(\mu_1, \nu_1)$ is $(G_{2(\mu_2,\nu_2)}, G_{1(\mu_1,\nu_1)})$-equivariant, $\varphi^*_{(\mu,\nu)/\mathcal{H}}$ is well-defined. We shall show that $\varphi^*_{(\mu,\nu)/\mathcal{H}}$ is symplectic with respect to the reduced symplectic forms, and $\mathcal{C}_{1(\mu_1,\nu_1)} = \varphi^*_{(\mu,\nu)/\mathcal{H}}(\mathcal{C}_{2(\mu_2,\nu_2)})$. In fact, since $\varphi^*: T^*Q_2 \to T^*Q_1$ is symplectic with respect to their magnetic symplectic forms, the map $(\varphi^*)^* : \Omega^2(T^*Q_1) \to \Omega^2(T^*Q_2)$ satisfies $(\varphi^*)^*\omega_{Q_1} = \omega_{Q_2}$. By (5.3), we have that $i^*_{(\mu_i,\nu_i)}\omega_{Q_i} = \pi^*_{(\mu_i,\nu_i)}\omega_{i(\mu_i,\nu_i)}$, $i = 1, 2$, and from the following commutative Diagram-2,

$$
\begin{array}{ccccc}
\Omega^2(T^*Q_1) & \xrightarrow{i^*_{(\mu_1,\nu_1)}} & \Omega^2(\mathbf{J}^{-1}_{Q_1}(\mu_1,\nu_1)) & \xleftarrow{\pi^*_{(\mu_1,\nu_1)}} & \Omega^2((T^*Q_1)_{(\mu_1,\nu)}) \\
{\scriptstyle (\varphi^*)^*}\downarrow & & {\scriptstyle (\varphi^*_{(\mu,\nu)})^*}\downarrow & & {\scriptstyle (\varphi^*_{(\mu,\nu)/\mathcal{H}})^*}\downarrow \\
\Omega^2(T^*Q_2) & \xrightarrow{i^*_{(\mu_2,\nu_2)}} & \Omega^2(\mathbf{J}^{-1}_{Q_2}(\mu_2,\nu_2)) & \xleftarrow{\pi^*_{(\mu_2,\nu_2)}} & \Omega^2((T^*Q_2)_{(\mu_2,\nu_2)})
\end{array}
$$

Diagram-2

we have that

$$\begin{aligned}
\pi^*_{(\mu_2,\nu_2)}\cdot(\varphi^*_{(\mu,\nu)/\mathcal{H}})^*\omega_{1(\mu_1,\nu_1)} &= (\varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot\pi_{(\mu_2,\nu_2)})^*\omega_{1(\mu_1,\nu_1)}\\
&= (\pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)})^*\omega_{1(\mu_1,\nu_1)}\\
&= (i^{-1}_{(\mu_1,\nu_1)}\cdot\varphi^*\cdot i_{(\mu_2,\nu_2)})^*\cdot\pi^*_{(\mu_1,\nu_1)}\omega_{1(\mu_1,\nu_1)}\\
&= i^*_{(\mu_2,\nu_2)}\cdot(\varphi^*)^*\cdot(i^{-1}_{(\mu_1,\nu_1)})^*\cdot i^*_{(\mu_1,\nu_1)}\omega_{Q_1}\\
&= i^*_{(\mu_2,\nu_2)}\cdot(\varphi^*)^*\omega_{Q_1}\\
&= i^*_{(\mu_2,\nu_2)}\omega_{Q_2} = \pi^*_{(\mu_2,\nu_2)}\omega_{2(\mu_2,\nu_2)}.
\end{aligned}$$

Notice that $\pi^*_{(\mu_2,\nu_2)}$ is a surjective, thus, $(\varphi^*_{(\mu,\nu)/\mathcal{H}})^*\omega_{1(\mu_1,\nu_1)} = \omega_{2(\mu_2,\nu_2)}$. Because by hypothesis $\mathcal{C}_i\subset\mathbf{J}^{-1}_{Q_i}(\mu_i,\nu_i)$, $\mathcal{C}_{i(\mu_i,\nu_i)} = \pi_{(\mu_i,\nu_i)}(\mathcal{C}_i),\ i=1,2$ and $\mathcal{C}_1 = \varphi^*_{(\mu,\nu)}(\mathcal{C}_2)$, we have that

$$\mathcal{C}_{1(\mu_1,\nu_1)} = \pi_{(\mu_1,\nu_1)}(\mathcal{C}_1) = \pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)}(\mathcal{C}_2) = \varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot\pi_{(\mu_2,\nu_2)}(\mathcal{C}_2) = \varphi^*_{(\mu,\nu)/\mathcal{H}}(\mathcal{C}_{2(\mu_2,\nu_2)}).$$

Next, from (4.3) and (5.5), we know that for $i=1,2$,

$$X_{(T^*Q_i,\mathcal{H},\omega_{Q_i},H_i,F_i,u_i)} = X_{H_i} + \mathrm{vlift}(F_i) + \mathrm{vlift}(u_i),$$

$$\begin{aligned}
&X_{(\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*,\tilde{\omega}^-_{\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*},h_{i(\mu_i,\nu_i)},f_{i(\mu_i,\nu_i)},u_{i(\mu_i,\nu_i)})}\\
&= X_{h_{i(\mu_i,\nu_i)}} + \mathrm{vlift}(f_{i(\mu_i,\nu_i)}) + \mathrm{vlift}(u_{i(\mu_i,\nu_i)}),
\end{aligned}$$

and from (5.6), we have that

$$\begin{aligned}
&X_{(\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*,\tilde{\omega}^-_{\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*},h_{i(\mu_i,\nu_i)},f_{i(\mu_i,\nu_i)},u_{i(\mu_i,\nu_i)})}\cdot\pi_{(\mu_i,\nu_i)}\\
&= T\pi_{(\mu_i,\nu_i)}\cdot X_{(T^*Q_i,\mathcal{H},\omega_{Q_i},H_i,F_i,u_i)}\cdot i_{(\mu_i,\nu_i)}.
\end{aligned}$$

Since H_i, F_i and $\mathcal{C}_i,\ i=1,2$, are all $\mathcal{H}$-invariant, and for $i=1,2$,

$$\begin{aligned}
h_{i(\mu_i,\nu_i)}\cdot\pi_{(\mu_i,\nu_i)} &= H_i\cdot i_{(\mu_i,\nu_i)},\\
f_{i(\mu_i,\nu_i)}\cdot\pi_{(\mu_i,\nu_i)} &= \pi_{(\mu_i,\nu_i)}\cdot F_i\cdot i_{(\mu_i,\nu_i)},\\
u_{i(\mu_i,\nu_i)}\cdot\pi_{(\mu_i,\nu_i)} &= \pi_{(\mu_i,\nu_i)}\cdot u_i\cdot i_{(\mu_i,\nu_i)}.
\end{aligned}$$

From the following commutative Diagram-3,

$$\begin{array}{ccccc}
TT^*Q_2 & \xleftarrow{Ti_{(\mu_2,\nu_2)}} & T\mathbf{J}^{-1}_{Q_2}(\mu_2,\nu_2) & \xrightarrow{T\pi_{(\mu_2,\nu_2)}} & T(T^*Q_2)_{(\mu_2,\nu_2)}\\
{\scriptstyle T\varphi^*}\downarrow & & {\scriptstyle T\varphi^*_{(\mu,\nu)}}\downarrow & & {\scriptstyle T\varphi^*_{(\mu,\nu)/\mathcal{H}}}\downarrow\\
TT^*Q_1 & \xleftarrow{Ti_{(\mu_1,\nu_1)}} & T\mathbf{J}^{-1}_{Q_1}(\mu_1,\nu_1) & \xrightarrow{T\pi_{(\mu_1,\nu_1)}} & T(T^*Q_1)_{(\mu_1,\nu_1)}
\end{array}$$

Diagram-3

we have that

$$T\varphi^*_{(\mu,\nu)/\mathcal{H}}X_{h_{2(\mu_2,\nu_2)}}\cdot\pi_{(\mu_1,\nu_1)} = T\pi_{(\mu_1,\nu_1)}\cdot T\varphi^*X_{H_2}\cdot i_{(\mu_1,\nu_1)},$$

$$\mathrm{vlift}(\varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot f_{2(\mu_2,\nu_2)}\cdot \varphi_{(\mu,\nu)/\mathcal{H}*})\cdot \pi_{(\mu_1,\nu_1)} = T\pi_{(\mu_1,\nu_1)}\cdot \mathrm{vlift}(\varphi^* F_2\varphi_*)\cdot i_{(\mu_1,\nu_1)},$$
$$\mathrm{vlift}(\varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot u_{2(\mu_2,\nu_2)}\cdot \varphi_{(\mu,\nu)/\mathcal{H}*})\cdot \pi_{(\mu_1,\nu_1)} = T\pi_{(\mu_1,\nu_1)}\cdot \mathrm{vlift}(\varphi^* u_2\varphi_*)\cdot i_{(\mu_1,\nu_1)},$$

where $\varphi_{(\mu,\nu)/\mathcal{H}*} = (\varphi^{-1})^*_{(\mu,\nu)/\mathcal{H}} : (T^*Q_1)_{(\mu_1,\nu_1)} \to (T^*Q_2)_{(\mu_2,\nu_2)}$. From the magnetic reducible controlled Hamiltonian matching condition MR-3 we have that

$$\begin{aligned}Im[(X_{h_{1(\mu_1,\nu_1)}} &+ \mathrm{vlift}(f_{1(\mu_1,\nu_1)}) - T\varphi^*_{(\mu,\nu)/\mathcal{H}}X_{h_{2(\mu_2,\nu_2)}}\\ &- \mathrm{vlift}(\varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot f_{2(\mu_2,\nu_2)}\cdot \varphi_{(\mu,\nu)/\mathcal{H}*})] \subset \mathrm{vlift}(\mathcal{C}_{1(\mu_1,\nu_1)}).\end{aligned} \tag{5.7}$$

So, from Definition 4.3 and Theorem 5.2 we get that

$$\begin{aligned}&(\mathcal{O}_{1(\mu_1,\nu_1)}\times V_1\times V_1^*, \tilde{\omega}^-_{\mathcal{O}_{1(\mu_1,\nu_1)}\times V_1\times V_1^*}, h_{1(\mu_1,\nu_1)}, f_{1(\mu_1,\nu_1)}, \mathcal{C}_{1(\mu_1,\nu_1)})\\ &\overset{RCH}{\sim} (\mathcal{O}_{2(\mu_2,\nu_2)}\times V_2\times V_2^*, \tilde{\omega}^-_{\mathcal{O}_{2(\mu_2,\nu_2)}\times V_2\times V_2^*}, h_{2(\mu_2,\nu_2)}, f_{2(\mu_2,\nu_2)}, \mathcal{C}_{2(\mu_2,\nu_2)}).\end{aligned}$$

Conversely, assume that the magnetic point reduced RCH systems $(\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*, \tilde{\omega}^-_{\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*}, h_{i(\mu_i,\nu_i)}, f_{i(\mu_i,\nu_i)}, \mathcal{C}_{i(\mu_i,\nu_i)})$, $i=1,2$, are RCH-equivalent. Then from Definition 4.3 and Theorem 5.2, we know that there exists a diffeomorphism $\varphi^*_{(\mu,\nu)/\mathcal{H}} : (T^*Q_2)_{(\mu_2,\nu_2)} \to (T^*Q_1)_{(\mu_1,\nu_1)}$, which is symplectic, and $\mathcal{C}_{1(\mu_1,\nu_1)} = \varphi^*_{(\mu,\nu)/\mathcal{H}}(\mathcal{C}_{2(\mu_2,\nu_2)})$, $(\mu_i,\nu_i)\in\eta_i^*$, $i=1,2$, and (5.7) holds. We can define a map $\varphi^*_{(\mu,\nu)} : \mathbf{J}^{-1}_{Q_2}(\mu_2,\nu_2)\to \mathbf{J}^{-1}_{Q_1}(\mu_1,\nu_1)$ such that $\pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)} = \varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot \pi_{(\mu_2,\nu_2)}$, and the map $\varphi^*: T^*Q_2\to T^*Q_1$ such that $\varphi^*\cdot i_{(\mu_2,\nu_2)} = i_{(\mu_1,\nu_1)}\cdot \varphi^*_{(\mu,\nu)}$, see the commutative Diagram-1, as well as a diffeomorphism $\varphi: Q_1\to Q_2$, whose cotangent lift is just $\varphi^*: T^*Q_2\to T^*Q_1$. From definition of $\varphi^*_{(\mu,\nu)}$, we know that $\varphi^*_{(\mu,\nu)}$ is $(G_{2(\mu_2,\nu_2)}, G_{1(\mu_1,\nu_1)})$-equivariant. In fact, for any $z_i\in \mathbf{J}^{-1}_{Q_i}(\mu_i,\nu_i)$, $(u_i,\alpha_i)\in G_{i(\mu_i,\nu_i)}$, $i=1,2$ such that $z_1=\varphi^*_{(\mu,\nu)}(z_2)$, and $[z_1]=\varphi^*_{(\mu,\nu)/\mathcal{H}}[z_2]$, then we have that

$$\begin{aligned}\pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)}(\Phi_{2(u_2,\alpha_2)}(z_2)) &= \pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)}((u_2,\alpha_2)\cdot z_2)\\ &= \varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot\pi_{(\mu_2,\nu_2)}((u_2,\alpha_2)\cdot z_2)\\ &= \varphi^*_{(\mu,\nu)/\mathcal{H}}[z_2] = [z_1] = \pi_{(\mu_1,\nu_1)}((u_1,\alpha_1)\cdot z_1)\\ &= \pi_{(\mu_1,\nu_1)}(\Phi_{1(u_1,\alpha_1)}(z_1)) = \pi_{(\mu_1,\nu_1)}\cdot\Phi_{1(u_1,\alpha_1)}\cdot\varphi^*_{(\mu,\nu)}(z_2).\end{aligned}$$

Since $\pi_{(\mu_1,\nu_1)}$ is surjective, hence, $\varphi^*_{(\mu,\nu)}\cdot\Phi_{2(u_2,\alpha_2)} = \Phi_{1(u_1,\alpha_1)}\cdot\varphi^*_{(\mu,\nu)}$. Moreover, we have

$$\pi_{(\mu_1,\nu_1)}(\mathcal{C}_1) = \mathcal{C}_{1(\mu_1,\nu_1)} = \varphi^*_{(\mu,\nu)/\mathcal{H}}(\mathcal{C}_{2(\mu_2,\nu_2)}) = \varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot\pi_{2(\mu_2,\nu_2)}(\mathcal{C}_2) = \pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)}(\mathcal{C}_2),$$

since $\mathcal{C}_i\subset \mathbf{J}^{-1}_{Q_i}(\mu_i,\nu_i), i=1,2$ and $\pi_{(\mu_1,\nu_1)}$ is surjective, then we get that $\mathcal{C}_1 = \varphi^*_{(\mu,\nu)}(\mathcal{C}_2)$. We shall show that φ^* is symplectic with respect to magnetic symplectic forms. Because $\varphi^*_{(\mu,\nu)/\mathcal{H}} : (T^*Q_2)_{(\mu_2,\nu_2)}\to (T^*Q_1)_{(\mu_1,\nu_1)}$ is symplectic with respect to reduced symplectic forms, the map $(\varphi^*_{(\mu,\nu)/\mathcal{H}})^* : \Omega^2((T^*Q_1)_{(\mu_1,\nu_1)})\to \Omega^2((T^*Q_2)_{(\mu_2,\nu_2)})$ satisfies $(\varphi^*_{(\mu,\nu)/\mathcal{H}})^*\omega_{1(\mu_1,\nu_1)} = \omega_{2(\mu_2,\nu_2)}$. By (5.3), we have that $i^*_{(\mu_i,\nu_i)}\omega_{Q_i} = \pi^*_{(\mu_i,\nu_i)}\omega_{i(\mu_i,\nu_i)}$, $i=1,2$, and from the commutative Diagram-2, we have that

$$\begin{aligned}i^*_{(\mu_2,\nu_2)}\omega_{Q_2} &= \pi^*_{(\mu_2,\nu_2)}\omega_{2(\mu_2,\nu_2)} = \pi^*_{(\mu_2,\nu_2)}\cdot(\varphi^*_{(\mu,\nu)/\mathcal{H}})^*\omega_{1(\mu_1,\nu_1)}\\ &= (\varphi^*_{(\mu,\nu)/\mathcal{H}}\cdot\pi_{(\mu_2,\nu_2)})^*\omega_{1(\mu_1,\nu_1)} = (\pi_{(\mu_1,\nu_1)}\cdot\varphi^*_{(\mu,\nu)})^*\omega_{1(\mu_1,\nu_1)}\\ &= (i^{-1}_{(\mu_1,\nu_1)}\cdot\varphi^*\cdot i_{(\mu_2,\nu_2)})^*\cdot\pi^*_{(\mu_1,\nu_1)}\omega_{1(\mu_1,\nu_1)}\\ &= i^*_{(\mu_2,\nu_2)}\cdot(\varphi^*)^*\cdot(i^{-1}_{(\mu_1,\nu_1)})^*\cdot i^*_{(\mu_1,\nu_1)}\omega_{Q_1} = i^*_{(\mu_2,\nu_2)}\cdot(\varphi^*)^*\omega_{Q_1}.\end{aligned}$$

Notice that $i^*_{(\mu_2,\nu_2)}$ is injective, thus, $\omega_{Q_2}=(\varphi^*)^*\omega_{Q_1}$. Since the vector field $X_{(T^*Q_i,\mathcal{H},\omega_{Q_i},H_i,F_i,u_i)}$ and $X_{(\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*,\tilde{\omega}^-_{\mathcal{O}_{i(\mu_i,\nu_i)}\times V_i\times V_i^*},h_{i(\mu_i,\nu_i)},f_{i(\mu_i,\nu_i)},u_{i(\mu_i,\nu_i)})}$ is $\pi_{(\mu_i,\nu_i)}$-related, $i=1,2$, and H_i, F_i and $\mathcal{C}_i,\ i=1,2$, are all $\mathcal{H}$-invariant, in the same way, from (5.7), we have that

$$Im[X_{H_1}+\text{vlift}(F_1)-T\varphi^*X_{H_2}-\text{vlift}(\varphi^*F_2\varphi_*)]\subset \text{vlift}(\mathcal{C}_1),$$

that is, the magnetic reducible controlled Hamiltonian matching condition MR-3 holds. Thus, from Definition 5.3 we get that

$$(T^*Q_1,\mathcal{H},\omega_{Q_1},H_1,F_1,\mathcal{C}_1)\overset{MRCH}{\sim}(T^*Q_2,\mathcal{H},\omega_{Q_2},H_2,F_2,\mathcal{C}_2). \qquad \blacksquare$$

6 Application: The Heisenberg Particle in a Magnetic Field

In this section, we consider the motion of a particle of mass m and charge e moving in the Heisenberg group $\mathcal{H}$ under the influence of a given magnetic field B, where B is a closed two-form on $\mathcal{H}$. The phase space of motion of the particle is the cotangent bundle $T^*\mathcal{H}$, which is trivialized locally as $\mathcal{H}\times\eta^*$ with the cotangent coordinates $(q^i,p_i),\ i=1,2,3$. The expressions of canonical symplectic form ω_0, the closed two-form B and the magnetic symplectic form ω_B on $T^*\mathcal{H}$ are given by

$$\omega_0=\sum_{i=1}^3\mathbf{d}q^i\wedge\mathbf{d}p_i,\qquad B=\sum_{i,j=1}^3 B_{ij}\mathbf{d}q^i\wedge\mathbf{d}q^j,\quad \mathbf{d}B=0,$$

$$\omega_B=\omega_0-\pi^*B=\sum_{i=1}^3\mathbf{d}q^i\wedge\mathbf{d}p_i-\sum_{i,j=1}^3 B_{ij}\mathbf{d}q^i\wedge\mathbf{d}q^j.$$

Here $q=(q^1,q^2,q^3)\in\mathcal{H}$ is the position of the particle in $\mathcal{H}$, and $p=(p_1,p_2,p_3)\in\eta^*$ is the momentum of the particle. Assume that there is a left-invariant metric $<,>_{\mathcal{H}}$ on the Heisenberg group $\mathcal{H}$. The Hamiltonian $H:T^*\mathcal{H}\to\mathbb{R}$ is given by the kinetic energy of the particle, that is,

$$H(q,p)=\frac{1}{2m}<p,p>_{\mathcal{H}}.$$

Note that the Hamiltonian does not dependent on the variable q and hence $\frac{\partial H}{\partial q^i}=0,\ i=1,2,3.$ From the (magnetic) Hamilton's equation $\mathbf{i}_{X_H}\omega_B=\mathbf{d}H$, we can get the Hamiltonian vector field as follows

$$X_H=\sum_{i=1}^3\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}+\frac{2e}{c}\sum_{i,j=1}^3 B_{ij}\frac{\partial H}{\partial p_j}\frac{\partial}{\partial p^i},$$

where c is the speed of light, and hence we obtain the equation of motion for the Heisenberg particle.

Moreover, we consider the magnetic potential $A:\mathcal{H}\to\eta^*$, which is an one-form on the Heisenberg group $\mathcal{H}$ and $B=\mathbf{d}A$. Then the fiber translation map $t_A:T^*\mathcal{H}\to T^*\mathcal{H},\ (q,p)\to(q,p+\frac{e}{c}A)$ can pull back the canonical symplectic form ω_0 of $T^*\mathcal{H}$ to the magnetic symplectic form ω_B, that is, $t_A^*\omega_0=\omega_0-\pi^*\mathbf{d}A=\omega_0-\pi^*B=\omega_B$, where $\pi:T^*\mathcal{H}\to\mathcal{H}$ is the natural projection. The modified Hamiltonian $H_A:T^*\mathcal{H}\to\mathbb{R}$ is given by

$$H_A(q,p-\frac{e}{c}A)=\frac{1}{2m}<p-\frac{e}{c}A,\ p-\frac{e}{c}A>_{\mathcal{H}}.$$

From the canonical Hamilton's equation $\mathbf{i}_{X_{H_A}}\omega_0 = \mathbf{d}H_A$, we can get the same Hamiltonian vector field, that is, $X_{H_A} = X_H$. In fact, from Marsden and Ratiu [9] we know why this is a general phenomenon by using the momentum shifting lemma.

On the other hand, we can also consider the magnetic term from the viewpoint of Kaluza-Klein construction. Assume that there is a Riemannian metric $<,>_Q$ on manifold $Q = \mathcal{H} \times S^1$, which is obtained by keeping the left-invariant metric $<,>_{\mathcal{H}}$ on $\mathcal{H}$ and the standard metric on S^1 and declaring $\mathcal{H}$ and S^1 orthogonal. The metric is called the Kaluza-Klein metric on Q. Note that the reduced Hamiltonian system is not the geodesic flow of the left-invariant metric $<,>_{\mathcal{H}}$, because of the presence of the magnetic term. However, the equation of motion of the Heisenberg particle in the magnetic field can be obtained by Legendre transformation and the reducing the geodesic flow of the Kaluza-Klein metric on $Q = \mathcal{H} \times S^1$. In the following we shall state how the magnetic term in the magnetic symplectic form $\omega_B = \omega_0 - \pi^* B$ is obtained by reduction from the Kaluza-Klein construction.

Assume that $Q = \mathcal{H} \times S^1$ with Lie group $G = S^1$ acting on Q, which only acts on the second factor. Since the infinitesimal generator of this action defined by $\xi \in \mathfrak{g} \cong \mathbb{R} = \mathrm{Lie}(S^1)$ has the expression $\xi_Q(q, \theta) = (q, \theta, 0, \xi)$, by using the left trivialization of T^*Q, that is, $T^*Q = \mathcal{H} \times S^1 \times \eta^* \times \mathbb{R}$, the momentum map $\mathbf{J}_Q : T^*Q = \mathcal{H} \times S^1 \times \eta^* \times \mathbb{R} \to \mathfrak{g}^* = \mathbb{R}$ is given by $\mathbf{J}_Q(q, \theta, p, \lambda)\xi = (p, \lambda) \cdot (0, \xi) = \lambda\xi$, that is, $\mathbf{J}_Q(q, \theta, p, \lambda) = \lambda$. In this case, the coadjoint action is trivial. For any $\mu \in \mathfrak{g}^* = \mathbb{R}$, we have that the isotropy group $G_\mu = S^1$, and its Lie algebra $\mathfrak{g}_\mu = \mathbb{R}$, and the one-form on Q, $\alpha_\mu = \lambda(A_{q_1}\mathbf{d}q_1 + A_{q_2}\mathbf{d}q_2 + A_{q_3}\mathbf{d}q_3 + \mathbf{d}\theta)$, where $\mathbf{d}\theta$ denotes the length one 1-form on S^1. Note that α_μ is S^1-invariant and its values are in $\mathbf{J}_Q^{-1}(\mu) = \{(q, \theta, p, \lambda) \in T^*Q \mid q \in \mathcal{H},\ \theta \in S^1,\ p \in \eta^*,\ \lambda \in \mathbb{R}\}$, and the exterior differential of α_μ equals $\beta_\mu = \mathbf{d}\alpha_\mu = \mu\mathbf{d}A = \mu B$. Thus, the closed 2-form β_μ on the base $Q_\mu = Q/G_\mu = (\mathcal{H} \times S^1)/S^1 = \mathcal{H}$, equals μB and hence the magnetic term, that is, the closed 2-form $B_\mu = \pi^*_{Q_\mu}\beta_\mu$, is also μB, since the map $\pi_{Q_\mu} : T^*Q_\mu = \mathcal{H} \times \eta^* \to Q_\mu = \mathcal{H}$ is the canonical projection. Therefore, from the cotangent bundle reduction theorem—embedding version, we know that the reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to $(T^*\mathcal{H}, \omega_B = \omega_0 - \mu B)$, which coincides with the phase space of Hamiltonian formulation of the Heisenberg particle in a magnetic field B. If we take that $\mu = e/c$, then the magnetic term in the magnetic symplectic form ω_B is the magnetic field B up to a factor.

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